A continuum theory of the isothermal flow of liquid helium II

By A. A. TOWNSEND
Emmanuel College, Cambridge

(Received 1 October 1960)

Recent work by Hall and Vinen has established that mutual friction between the normal and superfluid components of liquid helium II is caused by interactions between quantized vortex-lines and the normal fluid. If the mean separation of the vortex-lines is small compared with the channel width, the general character of the flow may not depend on the discrete nature of the lines except in so far as this is the cause of the mutual friction. Equations of motion are developed which refer to components of the velocity field with a scale large compared with the line separation, and these are used to discuss the nature of possible turbulent motions. Reasons are given for believing that isothermal flow is very similar to that of a Newtonian fluid, and the theory is developed for turbulent pressure flow along a channel and a circular pipe. The predicted variation of flow rate with pressure gradient is in good agreement with experimental measurements for Reynolds numbers (based on tube diameter and normal fluid viscosity) above 1400, and it is likely that turbulent flow can exist only above this critical Reynolds number. For Reynolds numbers which are not too small, the equations of motion apply to steady 'laminar' flow and these lead to a relation between flow rate and pressure gradient in reasonable agreement with experiment.

1. Introduction

The flow properties of liquid helium can be described in some detail by the two-fluid theory that assumes the liquid to be an intimate mixture of two components each capable of independent movement. One, the normal fluid, is a 'gas' composed of the thermal excitations (phonons and rotons) moving randomly through the whole liquid, and it may be accelerated by ordinary viscous forces of the gas-kinetic type. The superfluid contains the residual momentum and kinetic energy of the liquid and behaves as a perfect fluid at absolute zero without entropy or viscosity. Using the model in this form it is possible to describe quantitatively the flow of liquid helium II at not too high speeds, the essential properties being the inviscid behaviour of the superfluid, the free interpenetration of the two components and the thermodynamic mutual force which drives normal fluid down temperature gradients.

If the superfluid has zero viscosity, it should be impossible to set it in rotation from a condition of rest, but this can be done and the explanation of this anomaly is found in the theoretical result of Feynman (1955) that circulation is quantized in units of h/m (h is Planck's constant, m is the mass of a helium atom) and in the inference that vorticity is localized in vortex-lines of diameter comparable with the interatomic spacing. Once vorticity is introduced into the superfluid it can be increased by elongation of the lines in a kind of turbulent flow and, although most of the superfluid will be in irrotational motion, the motion induced by the vortex-lines will appear rotational on the macroscopic scale. Hall and Vinen have made an important extension by pointing out that the excitations of the normal fluid will be scattered in the velocity field of the vortex-lines and that this will cause a mutual friction if the vortex-lines are moving relative to the normal fluid. In a series of papers they have presented very convincing evidence in favour of the notion that mutual friction between the two fluids is entirely caused by this effect (for details, see Hall (1960)). The existence of a mutual friction provides the simplest explanation of the existence of a critical flow velocity above which the simple two-fluid theory is no longer valid.

The development of a flow in which all the vorticity is concentrated in discrete lines is a difficult mathematical problem, but many of the difficulties are avoided if we consider only those aspects of the flow that can be described in terms of averages over regions of space large enough to contain a substantial number of vortex-lines. The purpose of this paper is to construct such a continuum theory of the rotational flow of liquid helium II, including only those effects of concentrated vorticity that appear in these space averages. It is hoped that in this way knowledge of the turbulent flow of Newtonian fluids can be applied to the flow of liquid helium, although it is possible that significant effects take place on scales too small to be described by a continuum theory.

2. The continuum equations of motion

In detail, the motion of the superfluid is the result of a distribution of vortex-lines moving in their own velocity field and acted on by mutual forces, and a Fourier analysis of the whole velocity field would contain both components of small wave-number which represent streaming of large numbers of vortex-lines and components of large wave-number representing the velocity fields around individual vortices. If each line were split into two and the parts separated by half the previous spacing, considerable changes would be expected in the large wave-number components but very small changes in the components of small wave-number, so we identify the small wave-numbers with the continuum aspects of the flow and use them to define the part of the flow obeying equations of motion of classical type. The largest wave-number belonging to the continuum spectrum may be estimated by considering the dependence of the velocity field defined by

$$\mathbf{q}_a(\mathbf{x}) = (2\pi)^{-\frac{3}{2}}\,\delta^{-3}\int \mathbf{q}_t(\mathbf{x}+\mathbf{r})\,e^{-\frac{1}{2}(r^2/\delta^2)}\,d\mathbf{r}, \tag{2.1}$$

on the length δ. $\mathbf{q}_t(\mathbf{x}+\mathbf{r})$ is the true velocity field and $\mathbf{q}_a(\mathbf{x}+\mathbf{r})$ is roughly the average velocity in a spherical volume of radius $1{\cdot}55\delta$ surrounding the point $\mathbf{x}$.†

† Effective volumes and areas have been estimated by approximating the error function by a 'top-hat' function of equal volume or area.

If a moderately large number of vortex-lines pass through this volume, the average velocity will not be sensitive to the individual velocity fields of the lines but will be determined by their combined effect and by the effect of neighbouring lines. Supposing four to be a moderately large number, $\mathbf{q}_a(\mathbf{x})$ is essentially a continuum average if

$$\delta > \delta_0 = \left(\frac{2}{\pi L}\right)^{\frac{1}{2}}, \tag{2.2}$$

where L is the number of lines per unit area (or length per unit volume) and δ_0 may be used in (2.1) to define the continuum field of velocity. In terms of Fourier components, it is easily shown that only spectral components of wave-number less than

$$k_0 = 1{\cdot}55\left(\frac{\pi}{2}\right)^{\frac{1}{2}} L^{\frac{1}{2}} \approx 2\left(\frac{m\omega_s}{h}\right)^{\frac{1}{2}} \tag{2.3}$$

may be considered to belong to the continuum spectrum, i.e. the spectrum of the velocity field $\mathbf{q}_a(\mathbf{x})$.

Using these space averages to define flow variables for both components, the equations of motion may be obtained. Omitting terms involving changes in density, they are

$$\frac{\partial \mathbf{q}_s}{\partial t} + (\mathbf{q}_s . \mathrm{grad})\, \mathbf{q}_s = -\mathrm{grad}\,(p + p_d) + \frac{1}{\rho_s}\mathbf{F}_{sn} \tag{2.4}$$

for the superfluid, and

$$\frac{\partial \mathbf{q}_n}{\partial t} + (\mathbf{q}_n . \mathrm{grad})\, \mathbf{q}_n = -\mathrm{grad}\, p - \frac{1}{\rho_n}\mathbf{F}_{sn} + \frac{\nu_n}{\rho_n}\nabla^2 \mathbf{q}_n \tag{2.5}$$

for the normal fluid.† The quantity p_d in (2.4) represents the momentum flux due to the pressure and velocity fields of individual vortex-lines, and is of order $\omega_s h/m$.‡ The relatively high viscosity of the normal fluid makes it unlikely that the corresponding term is appreciable in the equation for its motion. The mutual force $\mathbf{F}_{sn}$ is the sum of the thermodynamic force proportional to the averaged temperature gradient and the volume average of the mutual forces arising from the scattering of excitations by the vortex-lines. The scattering calculations of Hall & Vinen (1956*b*) show that the mutual friction has two components, a drag force in the plane of the vortex-line and the relative velocity and a lift force normal to this plane, both proportional to the strength of the vortex-line and to the relative velocity. If the lines within the volume defined by δ_0 are jumbled and random in direction, the average of the lift forces will be zero and the average of the drag forces will be parallel to the averaged relative velocity and proportional to the length of line per unit volume. Allowing for the random orientation, the total mutual force is

$$\mathbf{F}_{sn} = S\,\mathrm{grad}\, T + B\rho_s\rho_n\omega_s(\mathbf{q}_n - \mathbf{q}_s), \tag{2.6}$$

† In these and all subsequent equations, the pressures, forces and viscosities are 'kinematic', i.e. they are the ordinary quantities divided by the total density of the fluid. ρ_s and ρ_n are then the mass fractions of the two components and $\rho_s + \rho_n = 1$.

‡ Hall (1960) has shown that the vortex-lines are in a state of tension, which is really the longitudinal effect of the radial pressure gradient necessary to maintain rotation of fluid particles about the line. The tension (in the form of the pressure component) is included in p_d.

where $\omega_s = Lh/m$ is the *vorticity of the superfluid*, and B is a function of temperature and nearly equal to one-third of the constant determined by measuring the attenuation of second-sound in rotating liquid helium (Hall & Vinen 1956*a*). This form for the mutual friction should be valid in turbulent flow but it needs modification if the vortex-lines are not randomly orientated (see § 5).

To these dynamical equations must be added the equations of state, of conservation of mass and of conservation of energy, but in isothermal flow at speeds small compared with either velocity of sound, changes in total density or in mass fraction of the components are expected to be negligible and we have

$$\operatorname{div} \mathbf{q}_s = \operatorname{div} \mathbf{q}_n = 0. \tag{2.7}$$

To make the set of equations formally complete, it is necessary to find some relation between the superfluid vorticity ω_s and the velocity field. If the vortex-lines had long-range order, i.e. adjacent lines nearly parallel, the superfluid vorticity equals the continuum vorticity, $|\operatorname{curl} \mathbf{q}_s|$, but this is unlikely to be true in general and it is necessary to discuss the processes of generation and destruction of superfluid vorticity.

If an ordinary Newtonian fluid is in turbulent flow, distributed vorticity is continually generated by stretching of the vortex-lines and is being destroyed at the same rate by the action of viscous forces. In a superfluid, vortex-lines can only be destroyed if they approach the wall or another, oppositely directed line within a few atomic diameters, but they can lose energy to the normal fluid without any need for such a close approach. The rate of energy loss depends on their density and their mean square velocity with respect to the normal fluid, which is also proportional to the density, and this kind of argument suggests that it is not necessary to discuss the mechanism of vortex-line destruction in detail. The problem may then be reduced to the part played by the vortex-lines in dissipating energy supplied to the flow, and the first step is to obtain equations for the kinetic energies of the turbulent fluctuations in the two components. For fully developed flow between parallel planes, these equations are

$$\overline{u_s v_s}\frac{\partial U_s}{\partial y} + \frac{\partial}{\partial y}(\overline{\tfrac{1}{2}q_s^2 v_s} + \overline{p v_s}) = B\rho_n \omega_s(\overline{\mathbf{q}_s \cdot \mathbf{q}_n} - \overline{q_s^2}) - \epsilon_s \tag{2.8}$$

for the superfluid, and

$$\overline{u_n v_n}\frac{\partial U_n}{\partial y} + \frac{\partial}{\partial y}(\overline{\tfrac{1}{2}q_n^2 v_n} + \overline{p v_n}) = B\rho_s \omega_s(\overline{\mathbf{q}_s \cdot \mathbf{q}_n} - \overline{q_n^2}) - \epsilon_n \tag{2.9}$$

for the normal fluid. In these equations, the Ox axis is parallel to the direction of flow and the Oy axis is at right-angles to the planes; the mean velocity has components $(U, 0, 0)$, the velocity fluctuation is (u, v, w) and $q^2 = u^2 + v^2 + w^2$. ϵ_n is the energy dissipation by viscous forces in the normal fluid and ϵ_s is the rate of loss of superfluid energy by transfer to scales of motion too small to contribute to the averaged velocity field. Similar forms may be derived for other kinds of flow.

An important difference between these equations and the corresponding equation for a Newtonian fluid is that mechanical energy can be lost to the system by mutual friction as well as by working against viscous retarding forces and that, while viscous losses hardly affect the large energy-containing eddies of the motion (for example, see Townsend 1956), this is not necessarily true of the mutual friction. Whether mutual friction affects the large eddies or not depends on the degree of correlation between the velocity fluctuations in the two fluids, and we consider the two extreme possibilities, (*a*) negligible correlation defined by

$$|\overline{\mathbf{q}_s . \mathbf{q}_n}| \ll (\overline{\mathbf{q}_s^2}\,\overline{\mathbf{q}_n^2})^{\frac{1}{2}}, \tag{2.10}$$

and (*b*) nearly perfect correlation defined by

$$\overline{(\mathbf{q}_s - \mathbf{q}_n)^2} \ll \frac{\rho_s \epsilon_s + \rho_n \epsilon_n}{B\rho_s \rho_n \omega_s}. \tag{2.11}$$

If the correlation is negligible, the mutual friction acts as a damping force but, if it is large, the mutual friction couples the motions together and the damping effect is relatively small.

In the energy equation for the superfluid fluctuations, the term

$$\frac{\partial}{\partial y}(\tfrac{1}{2}\overline{q_s^2 v_s} + \overline{p v_s})$$

represents transport of energy from one part of the flow to another and it is known to be comparatively small in channel flow (Laufer 1955). Omitting this term and assuming negligible correlation, we find that

$$\omega_s \leqslant \frac{|\overline{u_s v_s}|}{\overline{q_s^2}} \frac{1}{B\rho_n}\left|\frac{\partial U_s}{\partial y}\right| < \frac{1}{B\rho_n}\left|\frac{\partial U_s}{\partial y}\right| \tag{2.12}$$

and so the superfluid vorticity must be comparable with the vorticity of the mean flow. In turbulent flow, fluctuations of vorticity are typically large compared with the vorticity of the mean flow, and this suggests what the following analysis confirms, that independent velocity fluctuations in the two components are incompatible with isothermal turbulent flow and that the energy-containing eddies are almost perfectly correlated and coherent. The extent to which this coherence of motion extends to the small-scale components of the motion is considered in the next section.

3. Spectrum of coherent turbulent flow

If the velocity fields of the two components were identical except for a difference in mean velocity, the velocity fluctuation would satisfy the equation

$$\frac{\partial \mathbf{q}}{\partial t} + (\rho_s U_s + \rho_n U_n)\frac{\partial \mathbf{q}}{\partial x} + (\mathbf{q}.\mathrm{grad})\,\mathbf{q} = -\mathrm{grad}\,p + \nu_n \nabla^2 \mathbf{q}, \tag{3.1}$$

which is exactly the equation of motion for an ordinary fluid of the same viscosity moving with the mean flow velocity

$$U_m = \rho_s U_s + \rho_n U_n$$

in the same mean pressure gradient. This equation describes any perfectly coherent flow but it contains no terms involving thermomechanical forces. It follows that the turbulent motion observed during thermal flow of liquid helium is either of a scale too small to be described by a continuum theory or is fundamentally incoherent. Compare this equation with the equation of motion for the superfluid in the absence of thermomechanical forces:

$$\frac{\partial \mathbf{q}_s}{\partial t}+U_s\frac{\partial \mathbf{q}_s}{\partial x}+(\mathbf{q}_s.\mathrm{grad})\,\mathbf{q}_s = -\mathrm{grad}\,p+B\rho_n\omega_s(\mathbf{q}_n-\mathbf{q}_s). \tag{3.2}$$

On subtracting and equating $\mathbf{q}_s$ to $\mathbf{q}_n$ except in the mutual friction term, we obtain an approximate equation for the velocity difference in nearly coherent flow,

$$(U_s-U_n)\frac{\partial \mathbf{q}}{\partial x} = -B\omega_s(\mathbf{q}_s-\mathbf{q}_n)-\frac{\nu_n}{\rho_n}\nabla^2\mathbf{q}. \tag{3.3}$$

To this approximation, the equation is linear and, in terms of the Fourier coefficients of the velocity field, it is

$$-B\rho_n\omega_s(\mathbf{a}_s-\mathbf{a}_n) = [-\nu_n k^2+il(U_s-U_m)\rho_n]\,\mathbf{a}, \tag{3.4}$$

where $\mathbf{q} = \Sigma\mathbf{a}(\mathbf{k})\exp(i\mathbf{k}.\mathbf{x})$, or

$$\mathbf{a}_s-\mathbf{a}_n = \left(-il\frac{U_s-U_n}{B\omega_s}+\frac{\nu_n}{B\rho_n\omega_s}k^2\right)\mathbf{a}, \tag{3.5}$$

where l is the component in the direction of mean flow of the vector wave-number $\mathbf{k}$. Two conditions must be satisfied before essentially coherent motion is possible for the Fourier component of wave-number $\mathbf{k}$:

$$l \ll \frac{B\omega_s}{|U_s-U_n|} = l_c \quad \text{and} \quad k \ll \left(\frac{B\rho_n\omega_s}{\nu_n}\right)^{\frac{1}{2}} = k_1. \tag{3.6}$$

The first condition is inertial, requiring that the mutual force should be sufficient to supply the acceleration caused by relative motion of the superfluid and the pressure field, and to this approximation failure to satisfy the condition results in a difference in phase but none in magnitude. The second condition is independent of the relative velocity and requires a mutual force sufficient to balance the viscous stresses in the normal fluid. Failure to satisfy the second condition implies a greater magnitude of the Fourier component of the velocity fluctuation in the superfluid.

An essential process in any turbulent flow is the transfer of energy between eddies of different sizes, usually a transfer from large eddies to smaller ones, and it is usual to discuss this process in terms of the three-dimensional spectrum function $E(k)$, defined so that $E(k)\,dk$ is the fraction of the total energy $\frac{1}{2}\overline{\mathbf{q}^2}$ arising from Fourier components with magnitudes between k and $k+dk$ (Batchelor 1953). So long as the conditions (3.6) are satisfied, the motion is coherent and the low wave-number part of the spectrum is expected to be identical with the spectrum of flow in a Newtonian fluid and to have the same characteristics. For the present purpose, the relevant characteristics are (i) that viscous dissipation of turbulent energy is concentrated at wave-numbers near $k_s = \frac{1}{2}(\epsilon/\nu^3)^{\frac{1}{4}}$, where ϵ is the rate at which energy is lost by the larger eddies to the dissipative processes, (ii) that, if the wave-number is large compared with the wave-numbers characteristic of the energy-containing eddies, the motion at this wave-number is determined by the energy dissipation ϵ and by the kinematic viscosity ν_n, and (iii) that eddies with wave-numbers near k_s contain nearly all the vorticity. If these principles of local similarity are applied to a flow of liquid helium which is coherent at not too large wave-numbers, the statistical specification of the vorticity distribution in the superfluid depends on the rate of dissipation of mechanical energy, on the difference in mean velocity of the two fluids and on the physical properties of liquid helium. Dimensional considerations require then that the superfluid vorticity is given by

$$\omega_s^2 = \epsilon\nu_n^{-1} \times \text{function of } \left(\frac{\epsilon\nu_n}{(U_s - U_n)^4}, \frac{h}{m\nu_n}, B\rho_n\right). \tag{3.7}$$

To make this expression more precise, it is necessary to consider in more detail the transfer of energy from one Fourier component to another in the spectral range of coherent motion, in the range of incoherent continuum motion, and outside the continuum range. At first sight, there are a number of possibilities depending on the relative magnitudes of the wave-numbers defining the limits of coherent and of continuum motion, but in isothermal flow the limit to the coherent range is always set by the viscous condition. Then the existence of the velocity difference cannot affect the incoherent motion of the superfluid, and so the magnitude of the superfluid vorticity is independent of the velocity difference and equation (3.7) becomes

$$\omega_s^2 = \epsilon\nu_e^{-1}, \tag{3.8}$$

where ν_e/ν_n is a function of $h/(m\nu_n)$ and $B\rho_n$, i.e. of temperature.

The relative magnitudes that determine the ratio ν_e/ν_n are those of the limit of coherent motion k_1, the limit of continuum motion k_0 (defined by equation (2.3)), and the wave-number k_s near which the coherent spectrum loses energy by viscosity. Their ratios are

$$k_0/k_1 = 2\left(\frac{\nu_n m}{B\rho_n h}\right)^{\frac{1}{2}}, \quad k_s/k_1 = \frac{1}{2(B\rho_n)^{\frac{1}{2}}}\left(\frac{\nu_e}{\nu_n}\right)^{\frac{1}{2}},$$

and, on using the rough values $h/m\nu_n = 10$, $B\rho_n = 0{\cdot}05$, appropriate to a temperature near 1·4 °K, we find that there is a substantial spectral range of incoherent motion ($k_0/k_1 = 2{\cdot}8$). The continuum representation of the motion distinguishes between the part of the superfluid vorticity that is necessary to provide the averaged vorticity, curl $\mathbf{q}_s$, and the residual density of vortex-lines which is completely disordered. Since space averages are used in the definition of the continuum motion,

$$\omega_s^2 = \overline{(\text{curl}\, \mathbf{q}_s)^2} + \omega_d^2, \tag{3.9}$$

which defines $\omega_d = (h/m)L_d$ as the disordered component of the vorticity. Disorder in the distribution of vortex-lines causes each line to move relatively to its neighbours with a velocity of about $2h/md$, where d is the mean spacing. From the magnitude of k_0/k_1, we expect the normal fluid to be too viscous to move with the disordered motion of the vortex-lines and the rate of working against the mutual friction to be $B\rho_n L_d(2h/md)^2(h/m)$ per unit volume. Then the rate of energy dissipation by irregular motion of the vortex-lines is

$$\epsilon_s = \frac{\alpha B\rho_n}{\pi^2}\frac{h}{m}\omega_d^2, \tag{3.10}$$

where α is a constant of order one. The energy is the energy not dissipated by processes occurring in the continuum range of eddy sizes, and energy loss by mutual annihilation of vortex-lines is ignored.

As the details of the dissipation process for Newtonian fluids are not known in any detail, the magnitude of ν_e/ν_n will be left unspecified in the following sections. Notice that if the whole motion were coherent and all the energy were dissipated by viscous stresses, the superfluid vorticity would be $(\epsilon/\nu_n)^{\frac{1}{2}}$. Actually part of the total dissipation occurs in the incoherent range by mutual friction which is a less effective process than viscous dissipation and so more continuum vorticity will appear for the same continuum dissipation. In the disordered motion, too, more vorticity is necessary for the same energy dissipation and it seems a fair inference that ν_e is less than ν_n and more than $\alpha B\rho_n h/\pi^2 m$.

4. Pressure flow between parallel planes

In fully developed flow between parallel planes, mean values of the flow variables become independent of displacement parallel to the boundaries and functions only of y, the distance from one boundary. For this flow, the equations of mean motion are

$$\frac{\partial\overline{u_s v_s}}{\partial y} = \frac{\tau_0}{D} + B\rho_n\omega_s(U_n - U_s) \tag{4.1}$$

and

$$\frac{\partial\overline{u_n v_n}}{\partial y} = \frac{\tau_0}{D} + B\rho_s\omega_s(U_s - U_n) + \frac{\nu_n}{\rho_n}\frac{d^2U_n}{dy^2}, \tag{4.2}$$

where $2D$ is the width of the channel and the pressure gradient in the direction of flow is $-\tau_0/D$. From these equations may be obtained the equation for the mean mass flow,

$$\frac{\partial}{\partial y}[\rho_s\overline{u_s v_s} + \rho_n\overline{u_n v_n}] = \frac{\tau_0}{D} + \nu_n\frac{d^2U_n}{dy^2} \tag{4.3}$$

and the equation for the difference of mean velocity,

$$\frac{\partial}{\partial y}[\overline{u_n v_n} - \overline{u_s v_s}] = B\omega_s(U_s - U_n) + \frac{\nu_n}{\rho_n}\frac{d^2U_n}{dy^2}. \tag{4.4}$$

Equation (4.3) may be integrated to give the total stress,

$$\tau = \nu_n\frac{dU_n}{dy} - (\rho_s\overline{u_s v_s} + \rho_n\overline{u_n v_n}) = \tau_0(1 - y/D) \tag{4.5}$$

since τ_0 is the shear stress on either wall.

Arguments have already been given indicating that the motions of the large-scale eddies are nearly coherent, and if this is true the mass flow will be similar to that of an ordinary fluid in a channel of the same width and in the same pressure gradient except near the walls where the flow is unlikely to be turbulent. The validity of this notion will now be tested by calculating the amount of incoherence in the motion, but first it is useful to review the properties of channel flow as they have been established for Newtonian fluids. For values of $\tau_0^{\frac{1}{2}} y/\nu$ greater than 20, the distribution of mean velocity is of the form,

$$U = \frac{\tau_0^{\frac{1}{2}}}{K} F(y/D) + U_t, \tag{4.6}$$

where U_t is a velocity of translation that depends on details of the flow near the wall, the function $F(y/D)$ is characteristic of channel flow in general and K is an absolute constant about 0·41. This velocity distribution defines the rate of production of turbulent energy, $-\overline{uv}\,\partial U/\partial y$, and this rate equals the rate of energy dissipation in the most important part of the flow, that for which $y/D < 0{\cdot}2$. In this same region, $F(y/D) = \log y/D$ and so

$$\epsilon = -\overline{uv}\frac{\partial U}{\partial y} = \frac{\tau_0^{\frac{3}{2}}}{Ky} \tag{4.7}$$

and the scale of the energy-containing eddies is nearly proportional to distance from the wall (Townsend 1956).

It is now possible to estimate the magnitude of the departures from coherence in the liquid helium flow. The incoherence of the motions of the two components is given by equation (3.5), but the incoherence from inertial effects does not produce any inequality between $\overline{u_s v_s}$ and $\overline{u_n v_n}$, the first-order effect being a change of phase producing a small relative displacement of the velocity patterns. On the other hand, the effect of viscosity acting only on the normal fluid is to make velocities rather greater in the superfluid, and so

$$\frac{\overline{u_s v_s} - \overline{u_n v_n}}{\overline{uv}} = \frac{\nu_n^2 k_e^4}{B^2 \rho_n^2 \omega_s^2}, \tag{4.8}$$

where k_e is the average wave-number of the Fourier components contributing to the Reynolds stress. Substituting in equation (4.4), we obtain a relation between ω_s and $(U_s - U_n)$,

$$B\omega_s(U_s - U_n) = -\frac{d}{dy}\left[\frac{\nu_n^2 k_e^4 \tau}{B^2 \rho_n^2 \omega_s^2}\right] - \frac{\nu_n}{\rho_n}\frac{d^2 U_n}{dy^2}. \tag{4.9}$$

Using the relation (3.8), $\epsilon = \nu_e \omega_s^2$, and substituting for ϵ, k_e and dU_n/dy the values for a constant-stress layer in ordinary flow, viz.

$$k_e \propto y^{-1}, \quad \epsilon = \frac{\tau_0^{\frac{3}{2}}}{Ky}, \quad \frac{dU}{dy} = \frac{\tau_0^{\frac{1}{2}}}{Ky},$$

the ratio of the velocity difference to the scale of velocity variation $\tau_0^{\frac{1}{2}}$ is found to be

$$\frac{U_s - U_n}{\tau_0^{\frac{1}{2}}} = \frac{3K^{\frac{3}{2}}}{B^3 \rho_n^2}(k_e y)^4 \left(\frac{\nu_e}{\nu_n}\right)^{\frac{3}{2}} \left(\frac{\nu_n}{\tau_0^{\frac{1}{2}} y}\right)^{\frac{7}{2}} + \frac{1}{B\rho_n K^{\frac{1}{2}}}\left(\frac{\nu_e}{\nu_n}\right)^{\frac{1}{2}} \left(\frac{\nu_n}{\tau_0^{\frac{1}{2}} y}\right)^{\frac{3}{2}}. \tag{4.10}$$

It may now be confirmed that no appreciable incoherence exists in the fluctuations contributing to the energy and the Reynolds stress. The inertial condition for coherent motion becomes

$$B^2 \rho_n \frac{\nu_n}{\nu_e}\left(\frac{\tau_0^{\frac{1}{2}} y}{\nu_n}\right)^2 \gg k_e y = O(1), \tag{4.11}$$

and the viscous condition

$$B\rho_n \frac{\nu_n}{\nu_e}\left(\frac{\tau_0^{\frac{1}{2}} y}{\nu_n}\right)^{\frac{3}{2}} \gg (k_e y)^2 = O(1). \tag{4.12}$$

Both these conditions are satisfied in that part of the flow for which $\tau_0^{\frac{1}{2}} y/\nu_n$ is large, and this is the greater part of the channel if the Reynolds number of flow, $R = \tau_0^{\frac{1}{2}} D/\nu_n$, is large.

The assumption of the previous section that the upper limit to the coherent part of the spectrum is set by viscous rather than inertial effects may also be confirmed. Using equation (4.10), it may be shown that the ratio of the inertial cut-off to the viscous cut-off is

$$l_c/k_1 = B^{\frac{3}{2}} \rho_n^{\frac{1}{2}} K^{\frac{1}{4}} \left(\frac{\nu_n}{\nu_e}\right)^{\frac{1}{4}} \left(\frac{\tau_0^{\frac{1}{2}} y}{\nu_n}\right)^{\frac{5}{4}} \tag{4.13}$$

for large values of $\tau_0^{\frac{1}{2}} y/\nu_n$. At the same large values of $\tau_0^{\frac{1}{2}} y/\nu_n$, the mutual friction is

$$B\omega_s(U_s - U_n) = \frac{\tau_0}{D} \frac{D^2}{y^2} \frac{\nu_n}{\tau_0^{\frac{1}{2}} D}, \tag{4.14}$$

and it is small compared with the pressure gradient except very close to the wall.

In an ordinary fluid, a limit to fully turbulent flow is set by the damping action of the viscous forces which prevent an appreciable level of turbulent motion for values of $\tau_0^{\frac{1}{2}} y/\nu_n$ less than 12, and the additive constant in the velocity distribution (4.6) is determined by a condition of velocity continuity at the boundary between the fully turbulent flow and the effectively laminar flow next the wall. If the fluctuations in liquid helium were everywhere coherent, the same viscous forces would exist and impose the same limit to the region of fully turbulent flow, but the conditions for coherent flow, (4.10–12), show that it will not be possible for some values of $\tau_0^{\frac{1}{2}} y/\nu_n$ greater than 12 if $B\rho_n$ is small. Then the turbulent motion in this wall region is incoherent to some extent and the levels of turbulent intensity and Reynolds stress will be reduced because of added dissipation by mutual friction. To a rough approximation, we may use the viscous condition for coherence (4.12) to set a limit, y_v say, to the region of negligible Reynolds stress so that

$$\frac{\tau_0^{\frac{1}{2}} y_v}{\nu_n} = \chi\left(\frac{\nu_e}{B\rho_n \nu_n}\right)^{\frac{2}{3}}, \tag{4.15}$$

where χ is a constant. This can only be valid if y_v is much greater than the ordinary limit, $12\nu_n/\tau_0^{\frac{1}{2}}$. Applying the condition of velocity continuity at the edge of the region of negligible Reynolds stress, the universal velocity distribution (4.6) takes the form

$$U = \frac{\tau_0^{\frac{1}{2}}}{K}\left(\log \frac{\tau_0^{\frac{1}{2}} y}{\nu_n} + C\right), \tag{4.16}$$

where C is given by

$$C = K \frac{\tau_0^{\frac{1}{2}} y_v}{\nu_n} - \log \frac{\tau_0^{\frac{1}{2}} y_v}{\nu_n}. \tag{4.17}$$

For a Newtonian fluid, $\tau_0^{\frac{1}{2}} y_v/\nu_n \approx 12$ and $C \approx 2{\cdot}3$.

It has been shown that the mass flow velocity in the fully turbulent part of the flow has the same distribution as for an ordinary fluid except for a modified velocity of translation. Within the wall layer of negligible Reynolds stress, the flow is steady and analysis of this kind of flow (§ 5) shows that the velocity difference is nearly $2\nu_n/(3B\rho_n D)$ and the total flow in the wall layer is negligible for Reynolds numbers in the turbulent range. The volume flow may then be calculated from the universal velocity distribution as $Q_m = \frac{1}{D}\int_0^D U_m\,dy$, and is given by

$$\frac{Q_m}{\tau_0^{\frac{1}{2}}} = K^{-1}\left[\log R + C + \int_0^1 F(x)\,dx\right], \tag{4.18}$$

where $(R = \tau_0^{\frac{1}{2}} D/\nu_n)$. This relation between volume flow and pressure gradient is valid for flow along a tube of circular section with a suitably modified contribution from the integral of the distribution function $F(x)$.

The arguments set out here may be applied to any kind of isothermal turbulent flow and it seems that the continuum theory of turbulent flow in liquid helium II leads to the conclusion that its macroscopic structure is nearly indistinguishable from ordinary turbulent flow if the Reynolds number is large. This is not true for thermal flows in which the motion is essentially incoherent or for steady flows with vortex-lines.

5. Steady flow with vortex-lines

Many experimental measurements of flow resistance in tubes and channels have been made for Reynolds numbers between 10 and 50 and, although it is unlikely that the flow can be turbulent, the observed resistance indicates a sufficient density of vortex-lines for approximate validity of the continuum theory. It is natural to assume that the flow is steady and that the flow of the superfluid is retarded by mutual friction between the normal fluid and a steady stream of vortex-lines, generated perhaps near the channel entrance.† The continuum equations for steady flow between parallel boundaries are

$$\left.\begin{aligned} O &= +\frac{\tau_0}{D} + \tfrac{3}{2}B\rho_s\left|\frac{dU_s}{dy}\right|(U_s - U_n) + \frac{\nu_n}{\rho_n}\frac{d^2U_n}{dy^2}, \\ O &= \frac{\tau_0}{D} - \tfrac{3}{2}B\rho_n\left|\frac{dU_s}{dy}\right|(U_s - U_n), \end{aligned}\right\} \tag{5.1}$$

the factor $\frac{3}{2}$ appearing because the vortex-lines are all at right-angles to the flow direction and not randomly orientated as in turbulent flow.‡ Then,

$$U_n = \frac{\tau_0 y}{\nu_n}(1 - \tfrac{1}{2}y/D) \tag{5.2}$$

and the velocity difference between the superfluid and the normal fluid, $U_1 = U_s - U_n$, is given by

$$\tfrac{3}{2}B\rho_n U_1\left[\frac{dU_1}{dy} + \frac{\tau_0}{\nu_n}(1 - y/D)\right] = \frac{\tau_0}{D} \tag{5.3}$$

† The existence of such a flow is not certain.

‡ In steady flow the lift forces are all in the same direction and do not disappear when the averaging process is carried out, but their only effect is to induce a lateral pressure gradient.

with a boundary condition that depends on the nature of the interaction between the material of the wall and the liquid. The discussion of this boundary condition raises the same questions of the generation and continued existence of vortex-lines as arise from the existence of a critical velocity of flow. Without going into any detailed discussion it seems reasonable that there should be a layer of fluid next the wall not containing vortex-lines and moving with the critical velocity U_0 appropriate to its thickness d_0. Measurements of critical velocity suggest that these are related by

$$U_0 d_0 = \beta h/m, \tag{5.4}$$

where β is a number varying slowly with channel width. Consider now the forces acting on the layer of fluid next the wall and of thickness $d_0 + \frac{1}{2}d_1$, where d_1 is the mean separation of vortex-lines at the edge of the continuum flow. On the average, this layer is not accelerated and the pressure gradient is balanced by the mutual friction on the vortex-lines in the layer of thickness $\frac{1}{2}d_1$, so that

$$\tfrac{3}{2}B\rho_n \frac{h}{m}(U_0 - U_n) = \frac{\tau_0}{D}(d_0 + \tfrac{1}{2}d_1)\,d_1, \tag{5.5}$$

where U_n is the normal fluid velocity at distance d_0 from the wall. The mean separation of vortex-lines is related to the velocity gradient at distance d_0 from the wall by

$$\frac{dU_s}{dy} = d_1^{-2}\frac{h}{m},$$

so that these two equations provide a boundary condition.

Although solutions of equation (5.3) with this boundary condition could be computed, the essential features of the flow can be discovered by making suitable approximations. Near the wall, $y/D \ll 1$ and the solution is

$$\frac{\tau_0 y}{\nu_n} = A_0 - U_1 - \frac{\nu_n}{\frac{3}{2}B\rho_n D}\log\left[\frac{\frac{3}{2}B\rho_n U_1 D}{\nu_n} - 1\right], \tag{5.6}$$

where A_0 is a constant of integration determined by the boundary condition. This solution implies that the difference velocity approaches asymptotically the value $2\nu_n/(3B\rho_n D)$ but the final approach to the asymptotic solution requires more detail in the distribution of vorticity than can be provided by vortex-lines of finite strength. A continuum flow consistent with the physical requirements is one with the asymptotic difference flow to within a distance d_0 of the wall and a critical flow in this layer. The condition for the validity of the asymptotic solution is that $dU_1/dy \ll \tau_0/\nu_n$ or

$$R = \frac{\tau_0^{\frac{1}{2}} D}{\nu_n} \gg (\tfrac{3}{2}B\rho_n)^{-\frac{1}{2}}, \tag{5.7}$$

and in that part of the channel for which $(1 - y/D)^3 \gg (\frac{3}{2}B\rho_n R^2)^{-1}$, the solution may be continued as

$$U_1 = \frac{2\nu_n}{3B\rho_n D}(1 - y/D)^{-1}. \tag{5.8}$$

For still larger values of y/D, an approximate solution is

$$\tfrac{3}{4}B\rho_n U_1^2 = \frac{\tau_0 y}{D} + \text{constant}. \tag{5.9}$$

Using these approximate solutions in their ranges of validity, the difference flow in a channel may be calculated. It is

$$Q_1 D = \int_0^D U_1 dy = \frac{2\nu_n}{3B\rho_n}[\log(\tfrac{3}{2}B\rho_n R^2)^{\frac{1}{3}} + A'] + \beta\frac{h}{m}, \tag{5.10}$$

where A' depends on the transition between the regions of validity of (5.8) and (5.9) and is probably about one. In non-dimensional form, the volume flow is given by

$$\frac{Q_m D}{\nu_n} = \tfrac{1}{3}R^2 + \frac{2\rho_s}{3B\rho_n}[\log(\tfrac{3}{2}B\rho_n R^2)^{\frac{1}{3}} + A'] + \beta\rho_s\frac{h}{m\nu_n}. \tag{5.11}$$

These calculations have been made for flow between parallel planes but the extension to flow along a tube of circular cross-section is straightforward. In a tube of radius a, the wall stress is related to pressure gradient by

$$2\tau_0 = -a\frac{dP}{dx},$$

and the volume flow is given by

$$\frac{Q_m a}{\nu_n} = \tfrac{1}{4}R^2 + \frac{4\rho_s}{3B\rho_n} + 2\beta\rho_s\frac{h}{m\nu_n}, \tag{5.12}$$

where Q_m is the mean flow velocity of the whole liquid and the Reynolds number is defined by

$$R = \tau_0^{\frac{1}{2}} a/\nu_n.$$

These calculations refer to flow at comparatively large Reynolds numbers such that $\tfrac{3}{2}B\rho_n R^2 > 10$ and it is less easy to find explicit expressions for flow at lower Reynolds numbers within the continuum range. A very rough approximation is to neglect the motion of the normal fluid so that the solution of equation (5.3) is

$$U_1^2 = U_0^2 + \frac{\tau_0 y}{\tfrac{3}{4}B\rho_n D}, \tag{5.13}$$

and to use this solution for values of y greater than d_0, the distance from the wall of the first vortex-lines. An application of the boundary condition expressed in equations (5.4–5) leads to a slip velocity of

$$U_0 = \left[\frac{8\beta^2}{3B\rho_n}\frac{h}{m}\frac{\tau_0}{D}\right]^{\frac{1}{3}}, \tag{5.14}$$

if the normal fluid velocity is zero and the total volume flow may be calculated. It is given by

$$\frac{Q_m D}{\rho_s h/m} = \tfrac{4}{3}\beta^2\left[\left(1 - \frac{1}{2\beta} + (3\beta^4 B\rho_n)^{-\frac{1}{3}}\left(\frac{\tau_0^{\frac{1}{2}} D}{h/m}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} - 1\right] + \beta, \tag{5.15}$$

which has meaning only if

$$\frac{\tau_0^{\frac{1}{2}} D}{\nu_n} > (\tfrac{3}{8}\beta B\rho_n)^{\frac{1}{2}}\frac{h}{m\nu_n}$$

corresponding to $d_0 < D$. Experimental measurements of critical velocity and flow resistance in wide channels suggest that β is of order 10, so that this theory can only apply at Reynolds numbers greater than about three (for $B\rho_n \approx 0{\cdot}05$). On the other hand, neglect of the motion of the normal fluid is only possible if $U_1 D/\nu_n \ll 2/(3B\rho_n)$, corresponding to sub-critical flow rates at temperatures between 1·2 °K and the λ-point.

6. Comparison with measurements of flow resistance

The continuum theory of the flow of liquid helium II assumes that the macroscopic properties of the flow are determined by the pressure gradient, by the dimensions of the channel and by the properties of the liquid that can be

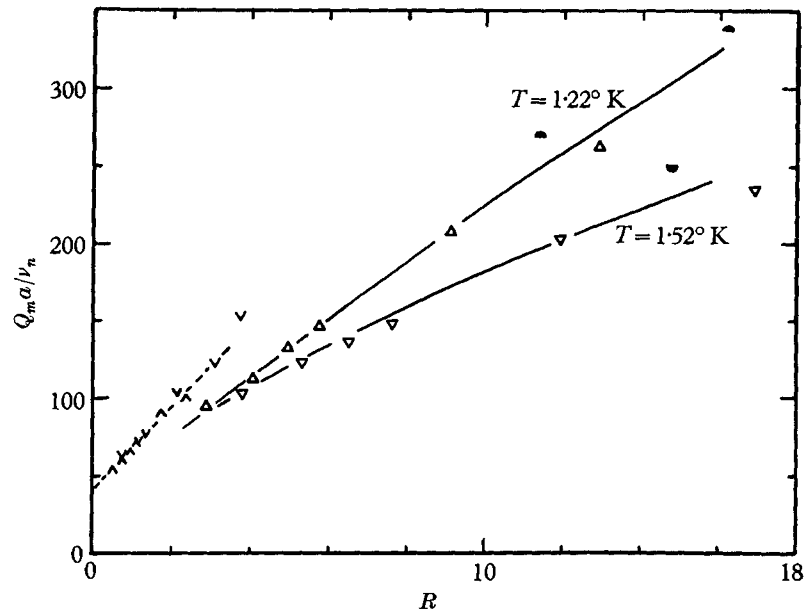


FIGURE 1. Reynolds number plot of flow measurements by Atkins (1951) at low Reynolds numbers.

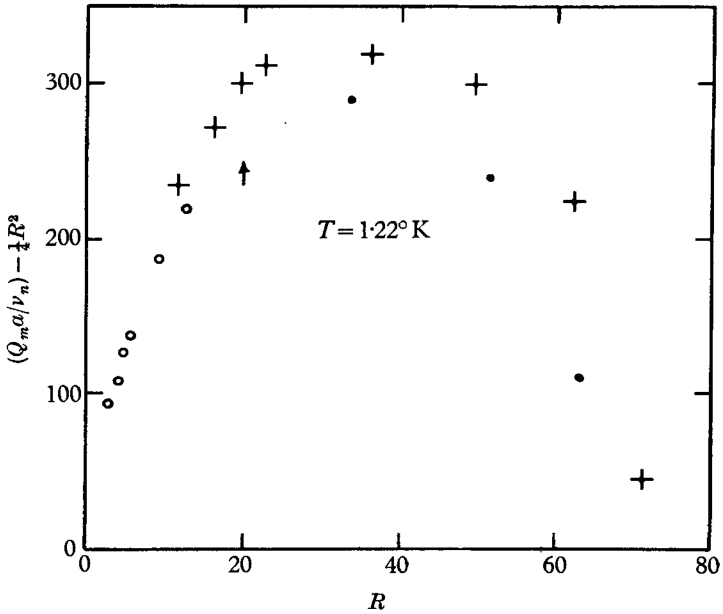
$a =$	$0{\cdot}131 \times 10^{-2}$ cm	$0{\cdot}408 \times 10^{-2}$ cm	$1{\cdot}02 \times 10^{-2}$ cm
$T = 1{\cdot}22$ °K	Λ	Δ	▲
$T = 1{\cdot}52$ °K	V	∇	▼

expressed by the quantities h/m, ν_n, ρ_n, B, which are either non-dimensional or have the dimensions of viscosity. Dimensional homogeneity then requires that flows should be dynamically similar if the Reynolds number and temperature are the same, and in particular that the volume flow velocity should be related to the pressure gradient by

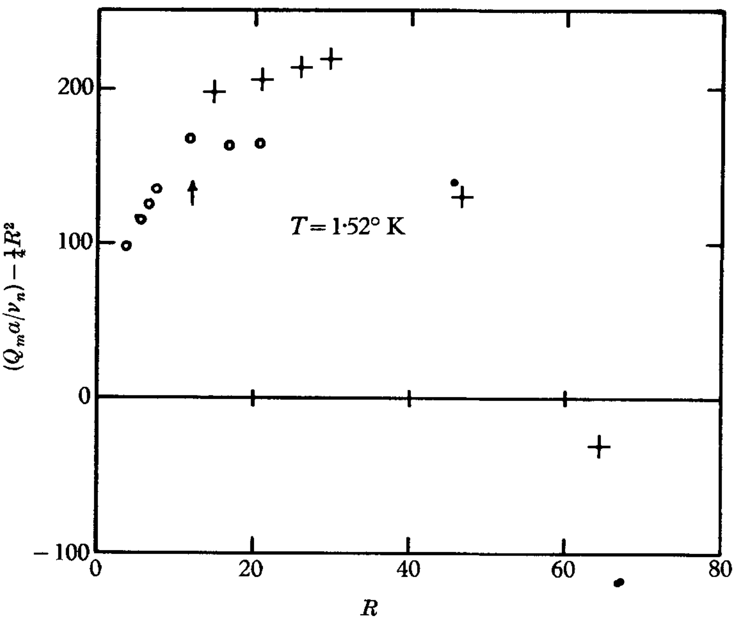
$$\frac{Q_m a}{\nu_n} = F\left(\frac{\tau_0^{\frac{1}{2}} a}{\nu_n}, \frac{h}{m\nu_n}, B, \rho_n\right), \tag{6.1}$$

where the function depends only on the flow geometry.† The only set of measurements that are suitable for a test of this prediction are some measurements of

† N.B. This is a test of dependence on the named quantities, not of the continuum theory as such.



(*a*)



(*b*)

FIGURE 2. Reynolds number plot of flow measurements by Atkins (1951) at moderate Reynolds numbers for comparison with equation (5.12). ○, $a = 0{\cdot}408 \times 10^{-2}$ cm; +, $a = 1{\cdot}02 \times 10^{-2}$ cm; ●, $a = 2{\cdot}2 \times 10^{-2}$ cm. Arrows indicate lower limit to validity of equation (5.12).

flow along capillary tubes by Atkins (1951). Using tubes of four different diameters and measuring the rate of flow at two different temperatures, his measurements covered a range of Reynolds numbers from one to 300, i.e. from outside the continuum range to well within the range for turbulent flow. Points taken from his mean curves have been plotted in non-dimensional form in figures 1–3, and comparatively good agreement with the functional form (6.1) is found. The principal cause of irregular behaviour appears to be caused by uncertainty in

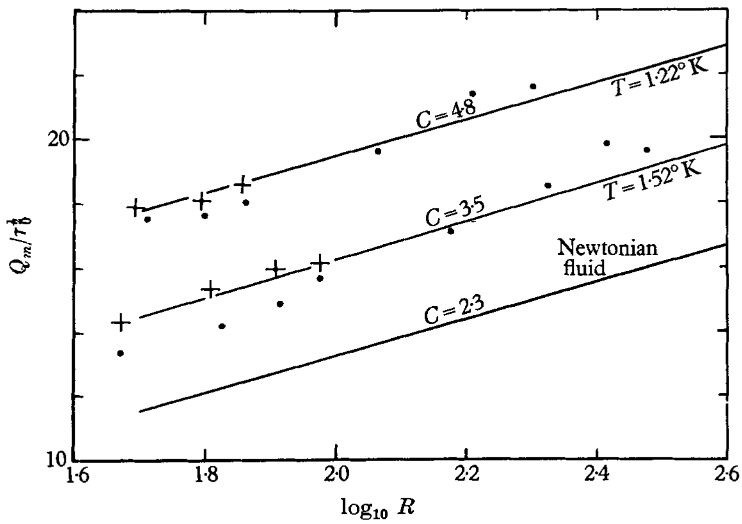


FIGURE 3. Reynolds number plot of flow measurements by Atkins (1951) at large Reynolds numbers for comparison with the logarithmic friction law (4.18). The lower line represents the mean of measurements for Newtonian fluids (Goldstein 1938).

+, $a = 1{\cdot}02 \times 10^{-2}$ cm; ●, $a = 2{\cdot}2 \times 10^{-2}$ cm.

the measured values of the tube diameter, as the discrepancies between points obtained with tubes of different diameters are of the same magnitude at both temperatures and show no consistent trend with tube diameter. On these results, there is no evidence that the absolute value of the diameter has any influence on the form of the relation (6.1) within the range of tube diameters

$$0{\cdot}262 - 4{\cdot}4 \times 10^{-2}\,\text{cm},$$

and if there is a length characteristic of the mechanism of mutual friction (e.g. the core diameter of a vortex-line) it is likely to be rather less than the smallest of these diameters.

Measurements at the smallest Reynolds numbers are shown in figure 1, most of them corresponding to average distances between vortex-lines larger than the tube radius. It is interesting that measurements with a Reynolds number less than four fall on the same curve at both temperatures (1·22° and 1·52 °K), but flow of the normal fluid must be very small in this range and a correlation of flow rate with normal fluid viscosity is very difficult to explain. Measurements at somewhat higher Reynolds numbers are compared with the steady flow theory of § 5 by plotting $(Q_m a/\nu_n) - \frac{1}{4}R^2$ against R (figure 2). Equation (5.12) requires that this difference flow rate should be independent of Reynolds number if the flow is steady and if $\frac{3}{2}B\rho_n R^2 > 10$, i.e. if R exceeds 12 at 1·52 °K or 20 at 1·22 °K, and this is approximately true for Reynolds numbers less than 50. Above this Reynolds number the apparent difference flow decreases rapidly, and it is natural to suppose that this abrupt change in the trend of the measurements corresponds to the growth of instabilities and the onset of turbulent motion. Confirmation is found by comparing the lower critical Reynolds number for flow of a Newtonian fluid which is 63. The difference and its sign might be expected from consideration of the effect of slipping of the superfluid at the wall on the condition for critical stability of disturbances of the steady flow.

In fully developed turbulent flow, the ratio $Q_m/\tau_0^{\frac{1}{2}}$ should be a linear function of $\log_{10} R$ with a slope of $K^{-1}\log_e 10 = 5{\cdot}66$ (see equation (4.18)), and this prediction is compared with the measurements in figure 3. Remembering that the division of the flow velocities by $\tau_0^{\frac{1}{2}}$ has removed the greater part of the variation with pressure gradient, it may be claimed that experiment and theory are in good agreement. The additive constant in the logarithmic velocity distribution appears in the mass-flow equation and these results show that the constant C is larger in liquid helium flow than in air or water flow and increases as the temperature is reduced. Reasons have already been given to expect this behaviour but the magnitude of the variation is less than is given by equation (4.15). This is not unreasonable as the effective thickness of the wall layer is no more than 1·4 times the similar layer in a Newtonian fluid and the necessary condition that the extent of this layer is controlled by the mutual friction and not by viscous damping is hardly satisfied.

More recently, Bhagat (1960) has made measurements of isothermal flow along a tube of radius 0·034 cm at temperatures in the range 1·3–1·7 °K, mostly at Reynolds numbers sufficiently high to permit turbulent flow. The measurements were made by observing the rate of change of level difference between two vessels connected by the tube and Bhagat found the variation of this rate with pressure difference could be accounted for if

$$Q_m = v_0 + \frac{a^2}{8\nu_{\text{eff.}}}\operatorname{grad} P, \tag{6.2}$$

where $\nu_{\text{eff.}}$ depended on the initial pressure gradient and the constant velocity, v_0, varied from one experiment to another, increasing with initial pressure gradient. Validity of equation (6.2) implies that the flow is not quasi-steady and that the friction at any stage is not solely determined by the instantaneous flow velocity but depends strongly on the turbulence generated in the initial rapid flow. However, it is characteristic of turbulent flows that the rate of adjustment to change is proportional to the rate of rotation of the eddies which must be large for fully developed flow in a small tube. So, although hysteresis and delay might be expected in the development of turbulent flow from an initial condition of very weak fluctuations, it is difficult to believe that this flow was not quasi-steady except at the very beginning. For the purposes of comparison with the theory and with the measurements of Atkins, it is unfortunate that Bhagat gives no details of the variation of v_0 and that it is difficult to decide the pressure difference appropriate to the measured value of $\nu_{\text{eff.}}$. In figure 4, the variations of $Q_m/r_0^{\frac{1}{2}}$ with R implied by equation (6.2) are shown for a number of initial pressure differences, using assumed values of v_0 that are consistent with the quoted order of magnitude, $2\,\text{cm}\,\text{sec}^{-1}$, and with the observed trend with pressure difference. All the curves fall near the line representing the logarithmic resistance law for 1·52 °K as inferred from Atkins's measurements, and the principal devia-

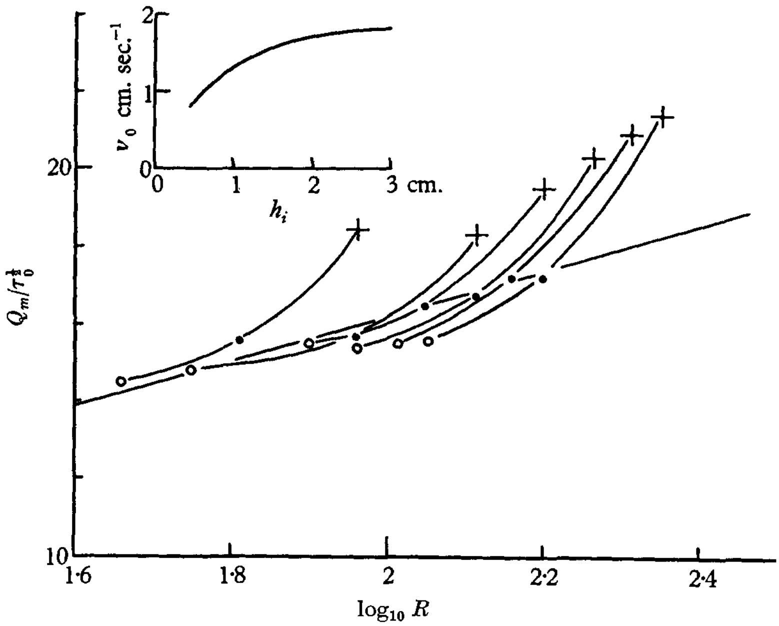


FIGURE 4. Variation of flow coefficient with Reynolds number according to Bhagat (1960) for various initial pressure gradients, assuming the values of v_0 shown in inset. The straight line is the logarithmic law for $C = 3{\cdot}5$. (The points +, ●, ○, refer to the initial gradient. one-half and one-quarter.)

tions correspond to abnormally large rates of flow while the turbulence level is increasing. It also appears from this diagram that the observed values of ν_{eff}. probably correspond to steady flow at about half the initial pressure gradient. Assuming this, a more direct comparison is possible, by comparing the observed values of $\nu_{\text{eff.}}/\nu_n$ with the values predicted by the logarithmic law,

$$\nu_{\text{eff.}}/\nu_n = \frac{R}{2(Q_m r_0^{-\frac{1}{2}} + K^{-1})}. \tag{6.3}$$

Reasonably good agreement is found, particularly at the lower temperatures (figure 5).

7. Discussion

This theory of the rotational flow of liquid helium depends on the demonstration by Hall and Vinen that resistance to flow of the superfluid is due to frictional forces between the normal fluid and the vortex-lines of the superfluid, and it attempts to obtain an approximate solution of the flow problem by considering only averages over volumes large enough to contain a substantial number of vortex-lines. This procedure is similar to the derivation of the Navier–Stokes equations from the kinetic theory of gases and within its proper limits this kind of approximation should be valid and useful. The important result of the calculations in this paper is that it is possible to assume that the motions of the normal

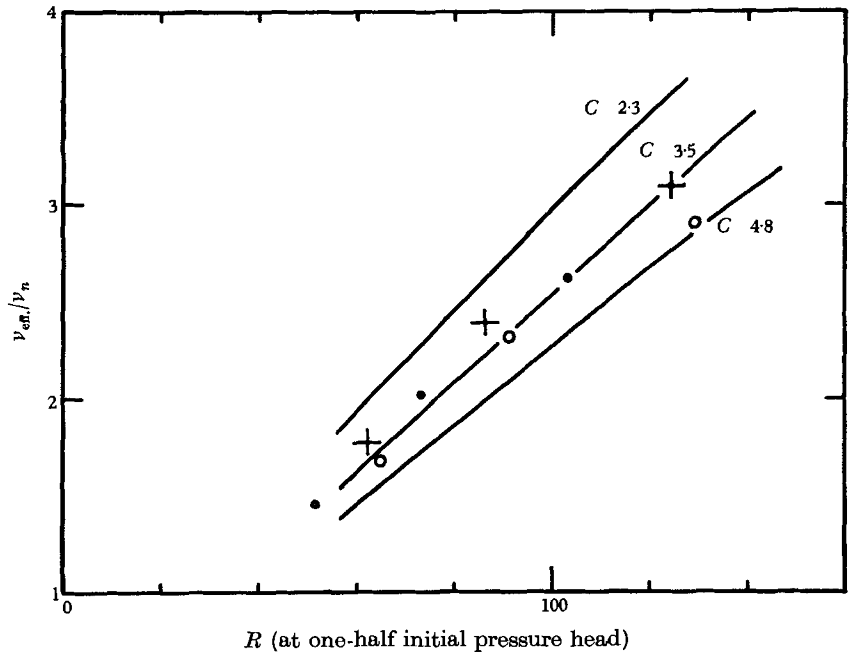


FIGURE 5. Variation of effective viscosity with Reynolds number calculated from half the initial pressure gradient (from Bhagat 1960). The lines are the variations predicted by the logarithmic law for several values of the additive constant C. ●, 1·3 °K; +, 1·5 °K; ○, 1·7 °K.

and the superfluid which contribute to the mean velocity and the Reynolds stress are almost perfectly coherent for turbulent flow in pipes and channels, the only region excepted from this being a thin layer next the walls. It is not very difficult to satisfy oneself that this will be true of any kind of isothermal flow and it appears that, if allowance is made for the peculiar properties of flow near a solid boundary, turbulent flow of liquid helium is very similar in its macroscopic aspects to turbulent flow of a Newtonian fluid.

The large differences between the flow of liquid helium and the flow of more ordinary fluids appear at Reynolds numbers less than 50, and I have attempted to describe the flow for the larger Reynolds numbers in this range by assuming the flow to be steady with vortex-lines moving in the general direction of flow in a quasi-regular array. The possibility of this kind of motion in liquid helium is not obvious and careful consideration of the flow details will be necessary before this assumption can be accepted without reservation. Meanwhile, it is interesting that the result that the excess of the superfluid flow over that of the normal fluid is, as the theory predicts, nearly independent of pressure gradient, and that the magnitude is consistent with the known magnitude of the friction constant B and with the assumption of a slip flow at the walls with the critical velocity appropriate to a channel of the same thickness. For example, measurements by Atkins at 1·52 °K give the product of slip velocity and layer thickness as

$$125 \pm 30 \times 10^{-4}\,\mathrm{cm^2\,sec^{-1}},$$

compared with values of $v_{s,c}d$ in the range $80\text{–}130 \times 10^{-4}\,\mathrm{cm^2\,sec^{-1}}$ for wide tubes.

Vinen (1957*a–d*) has shown conclusively that a heat current can induce something resembling turbulent motion in liquid helium and that the mutual friction is uniformly distributed across the channel. Simple considerations of the supply of energy to the 'turbulent' motion show that fully coherent motion can obtain no energy from the heat flow, and it follows that the important processes of the flow take place on scales comparable with the mean separation of the vortex-lines. Until a continuum theory of incoherent turbulence driven by thermomechanical forces is available, there must be some doubt whether the thermal flow can be usefully described by a continuum representation. However thermal flow is described, it seems certain that it is a completely different motion to isothermal flow and that comparisons of mutual frictions will not be profitable.

My interest in this subject was aroused by conversations with Dr H. E. Hall and Dr W. F. Vinen, and I am grateful for their help and criticism.

REFERENCES

Atkins, K. R. 1951 *Proc. Phys. Soc.* A, **64**, 833.

Atkins, K. R. 1959 *Liquid Helium.* Cambridge University Press.

Batchelor, G. K. 1953 *The Theory of Homogeneous Turbulence.* Cambridge University Press.

Bhagat, S. M. 1960 *Proc. Phys. Soc.* A, **75**, 303.

Feynman, R. P. 1955 *Progr. Low Temp. Phys.*, vol. I, chap. II. North Holland Publishing Co.

Goldstein, S. (ed.) 1938 *Modern Development in Fluid Dynamics*, vol. II. Oxford University Press.

Hall, H. E. 1960 *Phil. Mag. Suppl.* **9**, 89.

Hall, H. E. & Vinen, W. F. 1956*a* *Proc. Roy. Soc.* A, **238**, 204.

Hall, H. E. & Vinen, W. F. 1956*b* *Proc. Roy. Soc.* A, **238**, 215.

Laufer, J. 1955 *N.A.C.A. Rep.* no. 1174.

Townsend, A. A. 1956 *The Structure of Turbulent Shear Flow.* Cambridge University Press.

Vinen, W. F. 1957*a* *Proc. Roy. Soc.* A, **240**, 114.

Vinen, W. F. 1957*b* *Proc. Roy. Soc.* A, **240**, 128.

Vinen, W. F. 1957*c* *Proc. Roy. Soc.* A, **242**, 493.

Vinen, W. F. 1957*d* *Proc. Roy. Soc.* A, **243**, 400.